AD-A246 958

AFOSR-89-0261

THEORY AND APPLICATION OF RANDOM FIELDS

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Final Technical Report
1 January 1989 - 31 December 1991

Approved for public release.
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Prepared for:
Technion Research & Development Foundation;
European Office of Aerospace Research and Development, London, England;
Directorate of Mathematical Sciences, AFOSR, Washington.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION Unclassified	1b. RESTRICTIVE MARKINGS
2a. SECURITY CLASSIFICATION AUTHORITY	3. DISTRIBUTION/AVAILABILITY OF REPORT Approved for public release; distribution unlimited
2b. DECLASSIFICATION / DOWNGRADING SCHEDULE	
4. PERFORMING ORGANIZATION REPORT NUMBER(S)	5. MONITORING ORGANIZATION REPORT NUMBER(S)
6a. NAME OF PERFORMING ORGANIZATION Technion-Israel Inst of Technology 6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION European Office of Aerospace Research and Development
6c. ADDRESS (City, State, and ZIP Code) Faculty of Industrial Engineering & Management 32000 Haifa ISRAEL	7b. ADDRESS (City, State, and ZIP Code) Box 14 FPO New York 09510-0200
8a. NAME OF FUNDING/SPONSORING ORGANIZATION Air Force Office of Scientific Research 8b. OFFICE SYMBOL (If applicable) NM	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER AFOSR 89-0261
8c. ADDRESS (City, State, and ZIP Code) Bolling AFB, Washington DC 20332-6448	10. SOURCE OF FUNDING NUMBERS PROGRAM ELEMENT NO. 61102F / 61102F PROJECT NO. 2301 / 2304 TASK NO. D1 / A5 WORK UNIT ACCESSION NO. 014

11. TITLE (Include Security Classification)

Theory and Application of Random Fields

12. PERSONAL AUTHOR(S) R.J. Adler

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Year, Month, Day)	15. PAGE COUNT
Final	FROM 1 Jul 89 - 30 Dec 91	92-1-9	22

16. SUPPLEMENTARY NOTATION

17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB-GROUP	Random fields.

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

This report covers research performed under the three years of the grant.

Main progress has been in the areas of modelling problems related to measure valued diffusions, in the completion of a monograph on the properties and structure of Gaussian processes on general parameter spaces, and level crossing problems.

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT ☒ UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT. ☐ DTIC USERS	21. ABSTRACT SECURITY CLASSIFICATION Unclassified
22a. NAME OF RESPONSIBLE INDIVIDUAL Dr Arje Nachman	22b. TELEPHONE (Include Area Code) 202/767-4939 22c. OFFICE SYMBOL NM

DD FORM 1473, 84 MAR 83 APR edition may be used until exhausted. All other editions are obsolete.

SECURITY CLASSIFICATION OF THIS PAGE

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CONTENTS

1. INTRODUCTION

This report covers work carried out under the support of the AFOSR, under contract 89-0261, during the three year period January 1, 1989 until December 31, 1991. This three year project was a natural continuation and extension of similar work supported under contracts 83-0068, 84-0104, 85-0384 and 87-0298. The research program is centered on the study of various properties of random fields (stochastic processes whose "time" parameter is multi-dimensional) and includes the development of the requisite theoretical foundation to enable the application of these properties to specific modelling problems.

The last three years have been particularly successful, having seen the completion of a number of projects that have been going on for a while (in particular, the completion of a monograph on Gaussian processes) and the commencement of some major new directions of research. This makes the (hopefully temporary) lack of further support even more disappointing than it would normally be.

As one would expect, there will be a considerable overlap between this final report and the previous two annual reports.

2. ON RANDOM FIELDS

Random fields are stochastic processes, $X(t)$, whose parameter, t, varies over some general space rather than over the real line, in which case t is usually interpreted as time. The simplest of these occur when the parameter space is some multi-dimensional Euclidean space, and it is these fields that were at the centre of my research for a number of years. Of these, the most basic arise when the parameter space is the two-dimensional plane, so that we are dealing with some kind of random surface. When the parameter space is three-dimensional then we have a field (such as ore concentration in a geological site) that varies over space, while when the dimension increases to four we are generally dealing with space-time problems.

In space-time problems it is clear that one of the parameters is qualitatively different from the others, and so there is a natural tendency to denote this explicitly by writing the random field as

$$X(t, x),$$

where t is now one-dimensional time and x an N-dimensional space parameter. Of course, one could also write this as

$$X_t(x) \equiv X(t, x),$$

in which case one could consider X either as a random field on $(N+1)$-dimensional space, or, as a process in t, (one-dimensional) taking values $X_t(\cdot)$ in a space of N-dimensional random fields. In the latter case, since "time" has once again become one-dimensional, one can begin talking about Markov properties, diffusion processes (albeit infinite-dimensional from the point of view of their state space), etc.

Both of the above views of random fields will appear in this report, with differing methodologies and types of result appearing in each case. As well as this, random fields over very general parameter spaces that have no Euclidean structure at all will appear, with results of yet another nature.

Despite the different types of random fields that will appear in this report, there are two common threads running through it. One is of a theoretical nature, and is a result of my belief that despite the vast differences between many of these random fields, there is much to be gained by placing all of them within as common conceptual and mathematical frameworks as possible, and that this gain shows itself both in terms of developing intuition and in terms of developing streamlined proofs. The second thread is of a more practical nature, and comes from the fact that I believe that problems that have no possible application are unlikely to be even of theoretical interest. Thus, whereas most of the results in this report are not what one would normally call "applied", all the processes that I consider are related to real life modelling problems, and it is this, I believe, that ultimately makes them interesting and, often, difficult. Wherever it is possible to do so succinctly, the connection between theory and model will be described.

3. SPECIFIC PROJECTS

Much of my time over the past three years was dedicated towards the completion of two major projects. The first of these was the completion of a monograph on Gaussian processes described in the proposal. The second was the commencement of a major project in the study of measure and distribution valued stochastic processes, which were described there under the heading "random field valued processes". These two projects are described in some detail in the subsections (b) and (c) following. Over the past twelve months a good deal of effort has also gone into the more classical "level crossing problems", which are of more immediate applicability and closer in spirit to the work done under AFOSR support in previous years. One subsection is devoted to these.

A number of other problems that have also been looked are described in the final, "miscellaneous", subsection.

Yet others are not mentioned in the report at all, as can be gleaned by noting that the list of publications at the end of the report contains a goodly number of topics not discussed at all.

(A) MONOGRAPH ON GAUSSIAN PROCESSES:

A project that demanded a considerable amount of my time over the first two years of the grant period was the preparation of a monograph on the sample path continuity, boundedness, and extremal behaviour of Gaussian processes on general parameter spaces. This monograph appeared as Volume 12 of the prestigious Lecture Notes-Monograph Series of the Institute of Mathematical Statistics.

The main theme running through this work is that Gaussian processes should be studied in a general, unifying, framework, without particular emphasis being given to the specific geometric structure of their parameter spaces.

I presented series of lectures based on these notes in Sweden (February 1988, 12 lectures) Technion (second semester 1988/89, 26 lectures), and gave shorter sets of 2-3 lectures on a number of occasions. There were also groups at a number of institutions, including the University of California, Berkeley, and at the University of British Columbia, Vancouver, that held workshops based on my manuscript.

A good idea of what the manuscript is about can be gleaned from the following excerpt from the Preface.

"... on what these notes *are* meant to be, and what they are *not* meant to be.

They *are* meant to be an introduction to what I call the "modern" theory of sample path properties of Gaussian processes, where by "modern" I mean a theory based on concepts such as entropy and majorising measures. They are directed at an audience that has a reasonable probability background, at the level of any of the standard texts (Billingsley, Breiman, Chung, etc.). It also helps if the reader already knows something about Gaussian processes, since the modern treatment is very general and thus rather abstract, and it is a substantial help to one's understanding to have some concrete examples to hang the theory on. To help the novice get a feel for what we are talking about, Chapter 1 has a goodly collection of examples.

The main point of the modern theory is that the geometric structure of the parameter space of a Gaussian process has very little to do with its basic sample path properties. Thus, rather than having one literature treating Gaussian processes on the real line, another for multiparameter processes, yet another for function indexed processes, etc., there should be a way of treating all these processes at once. That this is in fact the case was noted by Dudley in the late sixties, and his development of the notion of entropy was meant to provide the right tool to handle the general theory.

While the concept of entropy turned out to be very useful, and in the hands of Dudley and Fernique lead to the development of necessary and sufficient conditions for the sample path continuity of stationary Gaussian processes, the general, non-stationary case remained beyond its reach. This case was finally solved when, in 1987, Talagrand showed how to use the notion of majorising measures to fully characterise the continuity problem for general Gaussian processes.

All of this would have been a topic of interest only for specialists, had it not been for the fact that on his way to solving the continuity problem Talagrand also showed us how to use many of our old tools in more efficient ways than we had been doing in the past.

It was in response to my desire to understand Talagrand's message clearly that these notes started to take form...

I rather hope that what is now before you will provide not only a generally accessible introduction to majorising measures and their ilk, but also to the general theory of continuity, boundedness, and suprema distributions for Gaussian processes.

Nevertheless, what these notes are *not* meant to provide is an encyclopædic and overly scholarly treatment of Gaussian sample path properties. I have chosen material on the basis of what interests me, in the hope that this will make it easier to pass on my interest to the reader. The choice of subject matter and of type of proof is therefore highly subjective."

The subjectiveness comes primarily from the fact that the examples and motivation of the monograph come from the area of set indexed empirical processes.

The contents page of the monograph is as follows:

AN INTRODUCTION TO CONTINUITY, EXTREMA, AND RELATED TOPICS FOR GENERAL GAUSSIAN PROCESSES

CONTENTS

(B) SUPER PROCESSES AND DENSITY PROCESSES:

The general area of infinite dimension diffusions is one of the most important and vital areas in Probability and Stochastic Processes today. For the first time since the introduction of interacting particle systems over a decade and a half ago, the theory of Markov and related processes has found a totally new family of processes that not only are of intrinsic interest, but require the kind of major extensions of an existing body of knowledge that rejuvinate mathematics and mathematicians. That this is the case can be seen from the calibre of the people now developing this area in the U.S. and Canada, and the flurry of conferences and sessions being dedicated to this topic. There is no doubt that this subject is about to blossom into the "bandwagon" of the early 1990's in Probability and Stochastic Processes, and the results and techniques that have been developed to date promise a very fruitful and rewarding ride.

One of the reasons that these processes were initially of particular interest to me is the fact, noted in Section 1, that they can be considered either as stochastic processes in one-dimensional time whose values are N-dimensional random fields, or as real valued $(N+1)$-dimensional random fields. Thus there was the natural challenge to see what could be done with them from the latter point of view. This challenge become even more interesting when, as time went by, it became clear that even the usual formulation was interesting from a purely random fields point of view.

To describe what these processes are, we start with a particle picture based on $N > 0$ initial particles which, at time zero, are independently distributed in $\Re^d$, $d \geq 1$, according to some finite measure m. We define a branching rate $\rho > 0$. Each of these N particles follows the path of independent copies of a Markov process Y, until time $t = \rho/N$.

At time ρ/N each particle, independently of the others, either dies or splits into two, with probability $\frac{1}{2}$ for each event. The individual particles in the new population then follow independent copies of Y in the interval $[\rho/N, 2\rho/N]$, and the pattern of alternating critical branching and spatial spreading continues until, with probability one, there are no particles left alive.

The process of interest for us is the measure valued Markov process

$$X_t^N(A) = \frac{\{\text{Number of particles in } A \text{ at time } t\}}{N}, \tag{1}$$

where $A \in \mathcal{B}^d$ = Borel sets in $\Re^d$. Note that, for fixed t and N, X_t^N is an atomic measure. Note also that if $\rho = \infty$ there is no branching occuring.

It is now well known that under very mild conditions on Y the sequence $\{X^N\}_{N\geq 1}$ converges weakly, on an appropriate Skorohod space, to a measure valued process which, when $\rho = 1$, is called the superprocess for Y. We shall be interested primarily in the cases in which Y is either an $\Re^d$-valued Brownian motion with infinitesimal generator Δ, or an $\Re^d$-valued symmetric stable process whose generator is the fractional Laplacian, $\Delta_\alpha := -(-\Delta)^{\alpha/2}$, $\alpha \in (0,2)$. When $\alpha = 2$ we define $\Delta_2 \equiv \Delta$. Then in either of these cases the corresponding superprocess, known, respectively, as a super Brownian motion or super stable process, can be defined as the unique solution of the following martingale problem:

For all $\phi \in C_b^2(\Re^d)$, the space of all bounded, continuous, $\Re$-valued functions on $\Re^d$ with continuous first and second order partial derivatives,

$$Z_t(\phi) = X_t(\phi) - m(\phi) - X_t(\Delta_\alpha \phi), \tag{2}$$

is a continuous square integrable martingale such that $Z_0 = 0$ and

$$\langle Z_\cdot(\phi)\rangle_t = \int_0^t X_s(\phi^2)\, ds. \tag{3}$$

(We have taken the obvious liberty here of denoting integration via $\int \phi(x)\,\mu(dx) = \mu(\phi)$ for a measure μ. Later, without further comment, we shall also write this as $\langle \phi, \mu \rangle$.)

When $\rho = \infty$ the situation is a little different, and X_t^N, as defined aboved, converges with probability one to the absolutely continuous measure whose density $x_t(u)$ satisfies the deterministic PDE $\partial x_t(u)/\partial t = \Delta_\alpha x_t(u)$ with initial condition $X_0 = m$. (This is just the strong law of large numbers.) In this case we look at the processes

$$\eta_t^N(\phi) = N^{1/2}\left\{X_t^N(\phi) - EX_t^N(\phi)\right\}, \tag{4}$$

where $\phi \in \mathcal{S}_d = \mathcal{S}(\Re^d)$, the Schwartz space of infinitely differentiable functions on $\Re^d$ decreasing rapidly at infinity, so that we think of η as a random (Schwartz) distribution rather than as a (signed) measure. The sequence $\{\eta^N\}_{N\geq 1}$ also converges weakly. Its limit is called the density process corresponding to Y, and it satisfies the the stochastic partial differential equation (SPDE)

$$\left(\frac{\partial \eta}{\partial t}\right)(\phi) = \eta(\Delta_\alpha \phi) + \sqrt{2} W_t(\Delta_{\alpha/2}\phi), \tag{5}$$

where W is a mean zero, space-time Gaussian process with covariance functional

$$E\, W(\phi_1 \times \psi_1) W(\phi_2 \times \psi_2) = \int_0^\infty \phi_1(t)\phi_2(t)\, dt \int_{\Re^d} \psi_1(x)\,\psi_2(x)\, dx. \tag{6}$$

with $\phi_i \in \mathcal{S}_1$ and $\psi_i \in \mathcal{S}_d$. In this case the limit process η is Gaussian.

The results that I have obtained for superprocesses and density processes involve their local and intersection local times.

The occupation time process for a superprocess X is a further measure valued process defined by

$$L_t(B) = \int_0^t X_s(B)\,ds. \tag{7}$$

This is clearly well defined for every Borel $B \in \Re^d$. The local time process is, formally, obtained by putting $B = \{x\}$ in (7). More precisely, let f be a positive C^∞ function supported in the unit d-ball, such that $\int_{\Re^d} f(x)dx = 1$. For each $\epsilon > 0$ set

$$f_\epsilon(x) \ = \epsilon^{-d} f(x/\epsilon), \tag{8}$$

and define

$$L_t^x \ = \ \lim_{\epsilon \to 0} \int_0^t X_s\big(f_\epsilon(\cdot - x)\big)\,ds. \tag{9}$$

The limit here, when it exists, may be taken in distribution or in $\mathcal{L}^2$, and is generally independent of the choice of f.

In order to be more precise about when the limit exists, we shall restrict ourselves to super Brownian and super stable processes. In these cases, not only can one establish existence results, but there are nice evolution equations describing the temporal evolution of the local time. To state these, we need to introduce the Green's functions

$$G_\alpha^\lambda(x,y) \ = \ \int_0^\infty e^{-\lambda t} p_t^\alpha(x,y)\,dt, \tag{10}$$

corresponding to the transition probabilities

$$p_t^\alpha(x,y) \ = \ p_t^\alpha(x-y) \ = \ \frac{1}{(2\pi)^d} \int_{\Re^d} \exp\big(-ip\cdot(x-y) - t\|p\|^\alpha\big)\,dp,$$

which, when $\alpha = 2$, is equivalent to Brownian transition probability

$$p_t^{(2)}(x,y) \ = p_t(x,y) \ = \ p_t(x-y) \ = \ \frac{1}{(4\pi t)^{d/2}} e^{-\|x-y\|^2/4t}.$$

The following result is in reference [11] in the lsit of publications following:

THEOREM. *Let X_t be either a super Brownian motion in dimension $d = 2,3$ or a super stable process in dimension $d < 2\alpha$, with $\alpha \in (\frac{1}{2}, 1]$ if $d = 1$. Then the local time L_t^x (9) is well defined as an $\mathcal{L}^2$ limit for each $x \in \Re^d$, is independent of the choice of f, and L_t^0 satisfies the following evolution equation for each $\lambda > 0$.*

$$L_t^0 \ = \ \langle G_\alpha^\lambda, X_0\rangle - \langle G_\alpha^\lambda, X_t\rangle + \lambda \int_0^t \langle G_\alpha^\lambda, X_s\rangle\,ds + \int_0^t \langle G_\alpha^\lambda, Z(ds)\rangle. \tag{11}$$

An identical equation holds for L_t^x if the function $G_\alpha^\lambda(\cdot)$ on the right hand side of the equation is replaced by $G_\alpha^\lambda(\cdot - x)$.

We now turn to the issue of (self)-intersection local time (ILT). At a heuristic level, this is a set indexed functional of the form

$$ILT(B) = \int_B dsdt \int_{\Re^d \times \Re^d} \delta(x-y)\, X_s(dx) X_t(dy), \tag{12}$$

where B is a finite rectangle in $[0,\infty)\times[0,\infty)$ and δ is the Dirac delta function. A more precise definition involves a limit as in the case of the regular local time. There are two qualitatively different cases that have to be considered in a rigorous formulation. The first, and by far the simpler case, arises when the set B does not intersect the diagonal $D = \{(t,t) : t \geq 0\}$. This case has been considered in detail in Dynkin for quite general superprocesses. The more difficult case, in which $B \cap D \neq \emptyset$, requires a renormalization argument. To set up the renormalization, note that the distributional equation

$$(-\Delta_\alpha + \lambda)u = \delta, \tag{14}$$

where δ is the Dirac delta function, is solved by $u = G_\alpha^\lambda$. That is, for every test function $\varphi \in \mathcal{S}_d$,

$$\int_{\Re^d} ((-\Delta_\alpha + \lambda) G_\alpha^\lambda)(x) \varphi(x) dx = \varphi(0).$$

Set $G_\epsilon^{\alpha\lambda} = G_\alpha^\lambda * f_\epsilon$, where "$*$" denotes convolution, and define a new approximate, renormalized ILT by setting, for every $\varphi \in \mathcal{S}_d$,

$$\begin{aligned} \gamma_\epsilon^\lambda(T,\varphi) &= \int_0^T dt \int_0^t ds \langle \varphi(x) f_\epsilon(x-y),\, X_s(dx) X_t(dy)\rangle \\ &\quad - \int_0^T \langle \varphi(x) G_\epsilon^{\alpha\lambda}(x-y),\, X_t(dx) X_t(dy)\rangle dt. \end{aligned} \tag{15}$$

Then the following result is in [10]:

THEOREM. *Let X_t be a super Brownian motion or super stable process, and let $\gamma_\epsilon(T,\varphi)$ be its approximate renormalized intersection local time as defined by (15). If $d = 4$ or 5 in the Brownian case, or $d/3 < \alpha \leq d/2$ in the stable case, then for all $\lambda > 0$, all $T \in (0,\infty)$ and all $\varphi \in \mathcal{S}_d$, $\gamma_\epsilon^\lambda(T,\varphi)$ converges in $\mathcal{L}^2$ to a finite limit $\gamma^\lambda(T,\varphi)$ as $\epsilon \to 0$ which we call the renormalized ILT of X_t, and which is independent of f. Furthermore, γ^λ has the following representation in terms of the process X_t and the martingale measure Z_t:*

$$\begin{aligned} \gamma^\lambda(T,\varphi) &= \int_0^T \int_{\Re^d} \left\{ \int_0^t \langle G_\alpha^\lambda(x-y)\varphi(x),\, X_s(dx)\rangle\, ds \right\} Z(dt,dx) \\ &\quad + \lambda \int_0^T dt \int_0^t ds\, \langle G_\alpha^\lambda(x-y)\varphi(x),\, X_s(dx) X_t(dy)\rangle \\ &\quad - \int_0^T \langle G_\alpha^\lambda(x-y)\varphi(x),\, X_t(dx) X_T(dy)\rangle\, dt. \end{aligned} \tag{16}$$

Except for the Brownian case in the plane, all parts of the theorem are also true for $\lambda = 0$. In this case, there are only two terms on the right hand side of (16).

The situation for density processes is somewhat different to that of superprocesses, since here the process is a distribution rather than a measure. In this case, we would like to use the following expression to define a new $\mathcal{S}'$ valued stochastic process, which we would like to serve as the intersection local time process in this case:

$$(17) \qquad \int_0^t du \int_0^t dv \int_{\Re^d} \int_{\Re^d} (\eta_u \times \eta_v)(\delta(x-y))\, \phi(x)\, dx dy.$$

Here $\eta_u \times \eta_v$ is the usual product of distributions.

When $d = 1$ and $\alpha > 1$ this is quite simple, for then the distribution $\int_0^t \eta_s \, ds$ has function form: i.e. There exists a local time $L_t(x)$ for η, so that (17) is given by $\int_{\Re} (L_t(x))^2 \phi(x)\, dx$. In general, however, one cannot make mathematical sense out of (17) without introducing a certain renormalisation. For this, unfortunately, we require some notation.

Let G_1, G_2 be a two zero mean, but otherwise completely general, Gaussian variables. We define their *Wick product* as $: G_1 G_2 := G_1 G_2 - EG_1 G_2$. Equipping $\mathcal{S}_d$ with the usual topology, let $\mathcal{A}_{2d}$ be the dense subset of $\mathcal{S}_{2d}$ made up of functions of the form

$$(19) \qquad \phi_N(x_1, x_2) \; = \; \sum_{i=1}^{N} \phi_i^{(1)}(x_1) \cdot \phi_i^{(2)}(x_2),$$

where $\phi_i^{(j)}(x) \in \mathcal{S}_d$. If η_1, η_2 are Gaussian distributions on $\mathcal{S}_2$ then we define the corresponding Wick product $: \eta_1 \times \eta_2 :$ on $\mathcal{S}_{2d}$ by setting

$$(: \eta_1 \times \eta_2 :)(\phi_N) \; = \; \sum_{i=1}^{N} : \eta_1(\phi_i^{(1)})\, \eta_2(\phi_i^{(2)}) :$$

for test functions of the form (19) in $\mathcal{A}_{2d}$ and then extending to all of $\mathcal{S}_{2d}$. That this is legitimate is standard fare in the theory of Gaussian distributions.

We are now in a position to make sense out of (17). For f, f_ϵ as before, $\phi \in \mathcal{S}_d$, and $t \geq 0$ set

$$\begin{aligned} \gamma_\epsilon(t, \phi) &= \gamma_\epsilon(f : t, \phi) \\ (20) \qquad &= \int_0^t dt_1 \int_0^t dt_2 \, (: \eta_{t_1} \times \eta_{t_2} :)\Big(\phi(x_1) f_\epsilon(x_2 - x_1)\Big). \end{aligned}$$

The following result is from [12], and has antecedents in [9]:

THEOREM. *Let η be the density process corresponding to either a Brownian motion ($\alpha = 2$) or index α symmetric stable process in $\Re^d$. Assume $d < 2\alpha$. Then the approximate ILT's $\gamma_\epsilon(f : t, \phi)$ converge in $\mathcal{L}^2$ as $\epsilon \to 0$. The limit random process $\gamma(t, \phi)$ is independent of the function f, and is called the ILT process for η_t.*

Furthermore, for all $\lambda > 0$, the limit γ (which itself is independent of λ) satisfies the following evolution equation:

$$\begin{aligned} \gamma^\lambda(t,\phi) &= 2\lambda \int_0^t du \int_0^u dv\,(:\eta_u \times \eta_v:)(G_\alpha^\lambda(x-y)\phi(y)) \\ &- 2\int_0^t du\,(:\eta_t \times \eta_u:)(G_\alpha^\lambda(x-y)\phi(y)) \\ &+ 2\int_0^t du\,(:\eta_u \times \eta_u:)(G_\alpha^\lambda(x-y)\phi(y)) \\ &+ 2\sqrt{2}\int_0^t \int_0^s \int_{\Re^d} \eta_u(\Delta_{\alpha/2} G_\alpha^\lambda(x-\cdot)\phi(\cdot))\,du\,W(dx,ds). \end{aligned} \tag{21}$$

Further work in this area is continuing apace. In particular, I am currently involved in trying to piece together deeper understanding of local and intersection local times.

For example, it follows from the above and related results that the super Brownian motion has a local time up to three dimensions, an intersection (over disjoint time sets) local time up to dimension seven, and a renormalised self-intersection local time up to dimension five.

On the other hand, the branching Brownian motions that, in the infinite density limit, provide a particle picture for the super Brownian motion have a local time in only one dimension, an intersection local time up to three dimensions, and a renormalisable self-intersection local time only in dimensions one and two.

What is of significant interest is the "dimension gap" between the particle picture and the superprocess, and a paper is under preparation that shows how to explain this gap in terms of weak convergence of functionals of the finite system to functionals on the superprocess.

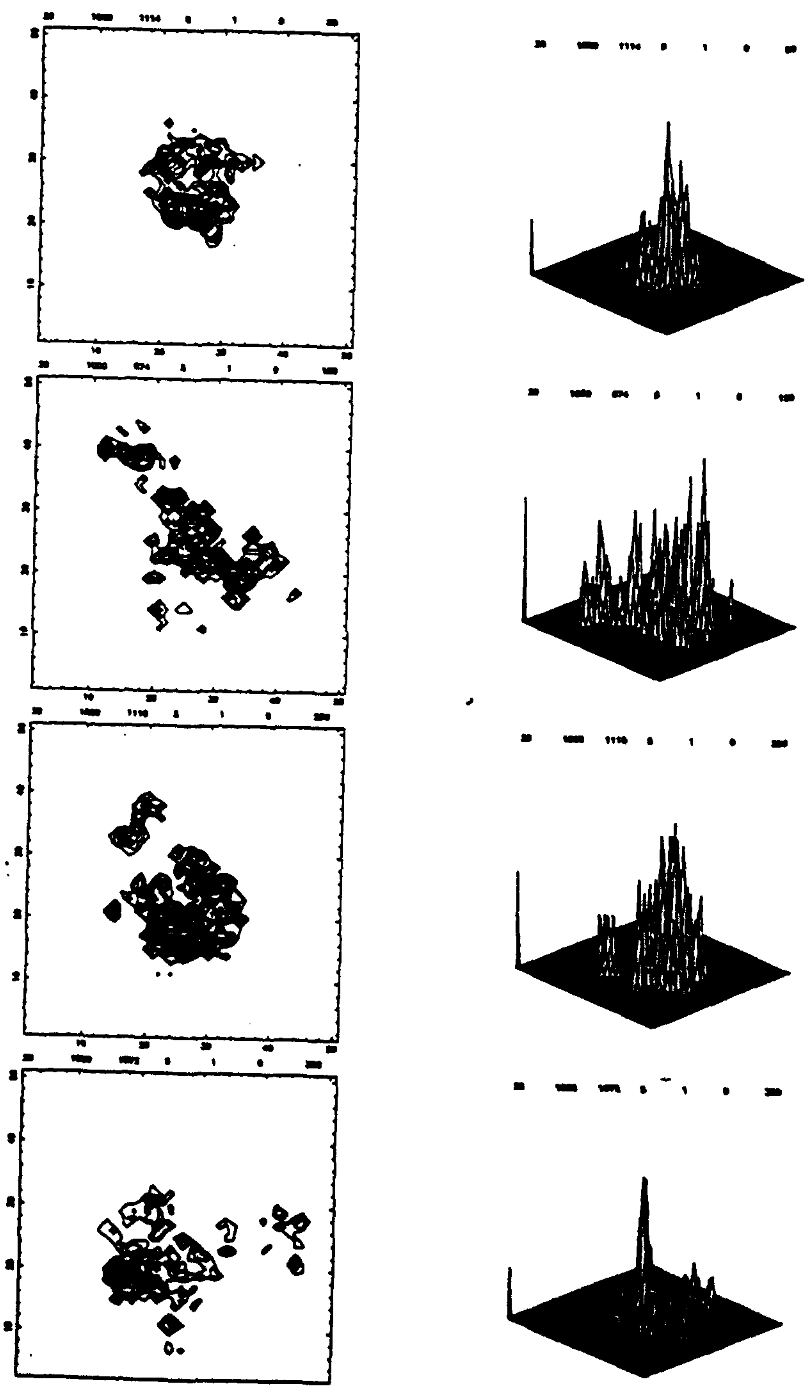
The problems raised here are related to the construction of measure valued processes with singular interactions and a general problem of the comparative richness of the class of $\mathcal{L}^2$ functionals on superprocesses when compared to those on the underlying $\Re^d$-valued Markov processes.

(c) VISUALISATION OF SUPER PROCESSES:

As is obvious from the preceeding section, superprocesses and their ilk are extremely technical objects. Furthermore, despite the significant effort currently being expended on the theoretical study of measure valued diffusions, they have yet to generate much interest among applied researchers. There are, I believe, two basic reasons for this. The first is that the literature is *extremely* technical, and beyond the reach of all but a relatively small group of researchers. This problem will certainly solve itself as expository papers and monographs start to appear. The second problem, however, is somewhat more basic, and lies in the fact that even fewer researchers have actually "seen" what these processes look like, and until one can show a user of stochastic processes what a "theoretical product" looks like, it is rather difficult to convince him to purchase it. Hence the interest in computer generation.

I received support from the Israel Academy of Sciences for a project of this nature, and have obtained an IBM RS6000 graphics workstation to carry out this project. The following page shows a picture of several time snapshots in the evolution of a measure valued process that began as a unit mass at the center of the square that spreads out in time. The height of the surface indicates the local mass of the measure, and the left hand pictures show contour lines. What is of interest is the uneven spread and "clumping" phenomenon. I have developed programs that show this evolution in real time and for a number of different processes.

It is important to note that computer generation of these rather complicated processes has more than a mere descriptive use. They have already led to the discovery of new properties of measure valued diffusions that have, to this point, escaped the attention of theoreticians. They will also, I hope, lead to real applications.



(D) Level Crossings:

Over the past twelve months I have been involved with two projects related to level crossings of stochastic processes.

The first, joint with Murad Taqqu and Gennady Samorodnitsky is about stationary, harmonisable, symmetric, α-stable ($S\alpha S$) stochastic processes $\{X_t,\, t \geq 0\}$. These processes have generated considerable interest over the past few years, primarily as a family of structurally Gaussian-like processes that provide good models for long tailed processes.

Let $C_u(T)$ denote the number of crossings of the level u by such a process in during the time interval $[0, T]$. Our primary interest lay in calculating $EC_u(T)$. Since X is assumed stationary, it is immediate that $EC_u(T) = TEC_u(1)$, and so we need only study $C_u := C_u(1)$.

The problem of having good information about EC_u is of major importance in terms of applying $S\alpha S$ processes in real life problems. There is probably no result as fundamental to the application of Gaussian processes as the famous Rice formula

$$EC_u = \frac{1}{\pi}\Big(\frac{-R''(0)}{R(0)}\Big)^{1/2} \exp\big(-u^2/2R(0)\big).$$

Without an analagous result for stable processes, many modelling applications of these processes cannot even begin.

It is reasonably obvious that it will be impossible to find a closed form expression for EC_u, since this (except for some very special cases) is even impossible for the much simpler stable densities. Thus bounds and approximations are the order of the day.

Our starting point is the the fact that stationary, harmonisable $S\alpha S$ processes can also be represented via the following infinite sum:

$$X(t) = \big(C_\alpha b_\alpha^{-1}\lambda_0\big)^{1/\alpha} \sum_{k=1}^{\infty} \Gamma_k^{-1/\alpha}\big(G_k^{(1)}\cos(t\Lambda_k) + G_k^{(2)}\sin(t\Lambda_k)\big).$$

The $\{G_k^{(i)}\}_k$, $i = 1, 2$, are independent sequences of i.i.d. standard normal variables. $\{\Gamma_k\}_k$ is a sequence of arrival times of a unit rate Poisson process, so that $\{\Gamma_{k+1} - \Gamma_k\}_k$ is a sequence of i.i.d. standard exponentials. $\{\Lambda_k\}_k$ is a sequence of i.i.d. random variables with distribution function $F(\lambda)/\lambda_0$ where $\lambda_0 := F(\infty)$. The four sequences are independent of one another. Finally, the constants are given by

$$\begin{aligned} C_\alpha &= \Big(\int_0^\infty x^{-\alpha}\sin x\, dx\Big)^{-1} \\ &= \begin{cases} (1-\alpha)\big(\Gamma(2-\alpha)\cos(\pi\alpha/2)\big)^{-1} & \text{if } \alpha \neq 1 \\ 2/\pi & \text{if } \alpha = 1. \end{cases} \\ b_\alpha &= 2^{\alpha/2}\Gamma(1+\alpha/2). \end{aligned}$$

The above representation is crucial for the study of EC_u. Note that if we condition on the $\{\Gamma_k\}$ and $\{\Lambda_k\}$ sequences, then what remains is a mean zero, stationary, Gaussian process with covariance function

$$R(t) = \left(\lambda_0 C_\alpha / b_\alpha\right)^{2/\alpha} \sum_{k=1}^{\infty} \Gamma_k^{-2/\alpha} \cos(t\Lambda_k).$$

Rice's formula gives us the precise form of the level crossing rate for the conditional Gaussian process as

$$E\{C_u \big| \{\Gamma_k\}, \{\Lambda_k\}\} = \frac{1}{\pi}\Big(\frac{\sum_{k=1}^{\infty} \Lambda_k^2 \Gamma_k^{-2/\alpha}}{\sum_{k=1}^{\infty} \Gamma_k^{-2/\alpha}}\Big)^{1/2} \exp\Big\{\frac{-u^2}{2\gamma_\alpha^2 \lambda_0^{2/\alpha} \sum_{k=1}^{\infty} \Gamma_k^{-2/\alpha}}\Big\},$$

where $\gamma_\alpha := (C_\alpha / b_\alpha)^{1/\alpha}$.

There seems to be no possibility of explicitly evaluating the remaining expectation here. There are, however, a number of paths that one can take that lead to useful results.

The first obvious path, which seems at first one of desperation, is to look at the behaviour of EC_u as $u \to \infty$. Given that more is known about the tails of stable random variables than about the central parts of their distributions, it is natural to hope that something can be done for this case. Our main result is:

THEOREM. *Assume $EC_0 < \infty$. Then*

$$\lim_{u \to \infty} u^\alpha EC_u = \frac{\lambda_1 C_\alpha}{\pi}.$$

It turns out that this result is of far more than theoretical interest, since the asymptotic formula inherent in the Theorem, viz.

$$EC_u \sim \frac{\lambda_1 C_\alpha}{\pi} u^{-\alpha},$$

provides a remarkably good approximation to EC_u once u is of the order of magnitude of the highest quartile of the distribution of X_t. This is shown by way of examples, including an explicit evaluation of EC_u for a stationary process and a combination of analytic and Monte Carlo techniques for some others.

We also have upper and lower bounds for EC_u that hold for all u and which, unlike previous results in the area, also hold for all α and are of the correct order of magnitude for large u.

In another piece of work with Tamar Gadrich, we have studied the structure of level crossings for ***non-stationary*** Gaussian processes $X(t)$, $-\infty < t < \infty$ with mean $m(t)$ and known but general covariance function. We also have explicit expressions for the

Slepian model process of nonstationary Gaussian processes following level crossings and local maxima and a detailed analysis of the high level case.

Two typical results are as follows:

(i) Given a u-upcrossing at time $t = 0$, $X(t)$ has the same finite-dimensional distributions as the process

$$\xi_u(t) = k(t) + m_t + A_t(Z - \dot{m}_0) + B_t(u - m_0)$$

where $k(t)$ is a zero mean continuously differentiable Gaussian process independent of Z and with a known, but complicated, covariance function. Both A and B are deterministic functions depending on the covariance function of X. The random variable Z has the density function:

$$f_Z(z) = \frac{\frac{z}{\sqrt{\gamma}}\phi\left(\frac{z-\omega}{\sqrt{\gamma}}\right)}{\omega\Phi\left(\frac{\omega}{\sqrt{\gamma}}\right) + \sqrt{\gamma}\phi\left(\frac{\omega}{\sqrt{\gamma}}\right)} I_{\{z\geq 0\}}$$

where:

$$\phi(x) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{x^2}{2}}$$

is the standard normal density function, and $\Phi(x)$ is the corresponding distribution function. γ and ω are known constatnts.

(ii) Given a local maximum of height u at $t = 0$, $X(t)$ has the same finite-dimensional distributions as the process

$$\xi_u^m(t) = L(t) + m_t + C_t(V - \ddot{m}_0) + D_t(u - m_0) - \dot{m}_0 E_t,$$

where $L(t)$ is a zero-mean continuously differentiable Gaussian process independent of V again with known covariance function. The random variable V has the density function:

$$f_V(v) = \frac{|v|\phi\left(\frac{\ell_1}{\sqrt{2\ell_2}} - \sqrt{2\ell_2}\ddot{m}_0 + \sqrt{2\ell_2}v\right)}{\left(\frac{\ell_1}{2^{3/2}\ell_2^{3/2}} - \frac{\ddot{m}_0}{\sqrt{2\ell_2}}\right)\Phi\left(\frac{\ell_1}{\sqrt{2\ell_2}} - \sqrt{2\ell_2}\ddot{m}_0\right) + \frac{1}{\sqrt{2\ell_2}}\phi\left(\frac{\ell_1}{\sqrt{2\ell_2}} - \sqrt{2\ell_2}\ddot{m}_0\right)} I_{\{v\leq 0\}}$$

Again C, D and E are deterministic functions depending on the covariance function of X and ℓ_2 is a known constant.

(E) MISCELLANEOUS:

Among the papers that appear in the list of publications below but do not relate to any of the topics discussed above, here are some details:

Paper [4] paper grew out of a joint project with Mike Marcus (City University of New York) and Joel Zinn (Texas A& M), on limit theorems for the local times of independent Markov processes, and the so called "isomorphism theorem" of Dynkin linking Gaussian and Markovian processes. While Marcus and Zinn were visiting the Technion, we began working on an empirical/Gaussian process problem, and got sidetracked on something else.

The problem we studied is described in the following excerpt from the abstract of [4]:

"Let $\{X_t, t \geq 0\}$ be an R^d-valued, symmetric, right Markov process with stationary transition density. Let $\{\hat{X}_t, t \geq 0\}$ denote the version of X_t killed at an exponential random time, independent of X_t. Associated with $\hat{X}_t$ is a Green's function $g(x,y)$, which we assume satisfies $0 < g(x,x) < \infty$ for all x and a local time $\{L_x, x \in R_d\}$. It follows from the so-called isomorphism theorem of Dynkin that L_x has continuous sample paths whenever $\{G(x), x \in R_d\}$, a Gaussian process with covariance $g(x,y)$, does. In this paper we use Dynkin's theorem to show that L_x satisfies the central limit theorem in the space of continuous functions on R^d if and only if $G(x)$ has continuous sample paths. This result strengthens a result of Adler and Epstein on the construction of the free field by means of a central limit theorem involving the local time, in the case when the local time is a point indexed process. In order to apply Dynkin's theorem the following result is obtained: The square at a continuous Gaussian process satisfies the central limit theorem in the space of continuous functions."

A joint project with Ron Pyke (University of Washington) led to papers [13,16]. The following is from the abstract of [13]:

"We study the uniform convergence of the quadratic variation of Gaussian processes, taken over large families of curves in the parameter space. A simple application of our main result shows that the quadratic variation of the Brownian sheet along all rays issuing from a point in $[0,1]^2$ converges uniformly (with probability one) as long as the meshes of the partitions defining the quadratic variation do not decrease too slowly. Another application shows that previous quadratic variation results for Gaussian processes on $[0,1]$ actually hold uniformly over large classes of partitioning sets."

One interesting comment to make regarding this work is that it is a nice example of how the general theory of Gaussian processes on abstract parameter spaces can be used to solve a very specific problem related to a very specific random process arising in Mathematical Statistics. Details are, of course, to be found in the paper.

Finally, there are two papers written by Nathalie Eisenbaum, who spent a year at the Technion following here doctoral studies at Paris VI, to where she has now returned. Dr. Eisenbaum was supported, in part by the grant, as a research assistant. Her papers deal with the interface between Gaussian and Markov processes, particularly via the "Isomorphism Theorem" of Dynkin, which establishes a close relation between the local time of a symmetric Markov process and an associated Gaussian process whose covariance function is given by the Green's function of the Markov process.

4. PUBLICATIONS UNDER THE GRANT

1. R. J. Adler, *An Introduction to Continuity, Extrema, and Related Topics for General Gaussian Processes*, (1990), vii + 160, IMS Lecture Notes-Monograph Series.
2. R. J. Adler, L. D. Brown and K-L Lu, Tests and confidence bands for bivariate cumulative distribution functions, *Communications in Statistics, Simulation and Computation*, 19, 1990, 25-36.
3. R. J. Adler, The net charge process for interacting, signed diffusions. *Annals of Probability*, 18, 1990, 602-625.
4. R. J. Adler, M. B. Marcus and J. Zinn, Central limit theorems for the local times of certain Markov processes and the squares of Gaussian processes. *Annals of Probability*, 18, 1126-1140, 1990.
5. N. Eisenbaum, The isomorphism theorem of Dynkin and Ray-Knight theorems, (16 pages, Submitted).
6. N. Eisenbaum, Additivity and strong infinite divisibility of Markov processes, (17 pages, Submitted).
7. R. J. Adler, Fluctuation theory for systems of signed and unsigned particles with interaction mechanisms based on intersection local times, *Advances in Appl. Probability*, 21, 1989, 334-356.
8. R. J. Adler, S. Cambanis and G. Samorodnitsky, On stable Markov processes, *Stochastic Processes and their Applications*, 34, 1990, 1-17.
9. R. J. Adler, R. Feldman and M. Lewin, Intersection local times for infinite systems of planar Brownian motions and the Brownian density process, *Annals of Probability*, 19, 192-220, 1991.
10. R. J. Adler and M. Lewin, An evolution equation for the intersection local times for super processes, *Stochastic Analysis*, eds. M.T. Barlow and N.H. Bingham, Cambridge University Press, 1991. 1-22.
11. R. J. Adler and M. Lewin, Local time and Tanaka formulae for super Brownian motion and super stable processes, *Stochastic Processes and their Applications*, 1991. (21 pages) In print.
12. R. J. Adler and J. S. Rosen, Intersection local times of all orders for Brownian and stable density processes – construction, renormalisation, and limit laws, *Annals of Probability*, 20, 1992. (48 pages) To appear.
13. R.J. Adler and R. Pyke, Uniform quadratic variation for Gaussian processes. (19 pages) Submitted.
14. T. Gadrich and R.J. Adler, Slepian models for non-stationary Gaussian processes. (12 pages) Submitted.

15. R.J. Adler, G. Samorodnitsky and T. Gadrich, The expected number of level crossings for stationary, harmonisable, symmetric, stable processes. (21 pages) Submitted.
16. R.J. Adler and R. Pyke, Brownian scan processes. In preparartion.
17. R.J. Adler, Superprocess local and intersection local times and their corresponding particle pictures. In preparartion.

5. CONFERENCES ATTENDED AND PROFESSIONAL VISITS

Conferences.

1. Math Sciences Institute Workshop on Markov Processes in Functional Spaces, Cornell University, May 14–16.
2. The 18th Conference on Stochastic Processes and Their Applications; Madison, Wisconsin, June 25–July 1.
3. July, 1990: London Mathematical Society Durham Symposium: Stochastic Analysis, Durham, England. *Intersection local time for distribution and measure valued processes*, (Invited lecture).
4. August, 1990: Second World Congress of the Bernoulli Society, Uppsala, Sweden. *Random Fields* (Session organiser).
5. The 20th Conference on Stochastic Processes and Their Applications; Nahariya, Israel, June 9–14, 1991. (Chairman of organising committee.)

University visits.

1. Department of Mathematics and Statistics, Carleton University, Ottowa, Canada. January 22 - Febraury 19, 1989. (Don Dawson, Miklos Csorgo).
2. Department of Statistics, University of California, Berkeley. February 19 - February 26, 1989. (Raisa Epstein, David Donoho).
3. Department of Operations Research, Cornell University, Ithaca. May 11 - May 17, 1989. (Gennady Samorodnitsky, Sidney Resnick).
4. Department of Statistics, University of Rome, Italy. June 14 - June 18, 1989. (Bruno Bassan, Arnoldo Frigessi, Enzo Orsinger).
5. Department of Mathematics, University of Washington, Seattle. September 8 - September 30, 1989. (Ron Pyke).
6. Department of Mathematics, University of British Columbia, Vancouver, September 19 - September 20, 1989. (Ed Perkins, John Walsh).
7. Department of Applied Probability and Statistics, University of California, Santa Barbara. February, 1990, July, 1991. (Raisa Epstein Feldman, Zari Rachev).

Travel costs were met by a combination of AFOSR and Technion funds. When visiting universities rather than attending conferences, local costs were generally met by the host institution.